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Stress Dependency of the Thermoelastic and Piezoelectric Coefficients

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Introduction

THE history of the thermomechanical coupling existing in solid materials goes back to Gough[1]; a concise review of the subject was given by Bert and Fu.[2] The purpose of the present Note is to extend this work to include the piezoelectric effect as well as the thermoelastic effect. An analysis based on fundamental mechanics and thermodynamics is presented for a solid in which the elastic and piezoelectric coefficients are temperature dependent. It is proven that, as a consequence of the temperature dependency, the thermal expansion and piezoelectric coefficients must be stress dependent.

Constitutive Relations

Using the Einstein summation convention, one may express the constitutive relations of a piezothermoelastic material as[3,4]

$$\sigma_{ij} = C_{ijkl}^{E,T}(T)\varepsilon_{kl} - e_{ijm}^{T}(T)E_m - \left[\lambda_{ij}^{E,T}(T)\right](T - T_0) \tag{1}$$

$$D_m = e_{mij}^{T}(T)\varepsilon_{ij} + \tilde{\varepsilon}_{mk}^{\varepsilon,T}(T)E_k + \left[p_m^{\varepsilon,T}(T)\right](T - T_0) \tag{2}$$

Here $C_{ijkl}^{E,T}$ are the elastic stiffnesses, D_m are electric displacements, E_m (or E_k) are electric-field components, e_{mij}^{T} are the elements of the piezoelectric tensor, $p_m^{\varepsilon,T}$ are pyroelectric constants, T is temperature (with a reference temperature T_0), ε_{ij} are engineering strains, $\tilde{\varepsilon}_{mk}^{\varepsilon,T}$ are the electric permittivities, $\lambda_{ij}^{E,T}$ are thermoelastic coefficients, and σ_{ij} are the stresses. The superscripts indicate that the main symbols' physical quantities are measured at constant levels of the superscript factors. For example, the elastic stiffnesses are measured at a constant electric field and constant temperature. Hereafter, the superscripts are omitted for clarity. The third constitutive relation[4] that is omitted here defined the changes of entropy in terms of independent variables, i.e., strain, electric field, and temperature. Following the common practice, this equation is omitted in the subsequent analysis because it is not concerned with a heat exchange process. The purpose of the analysis is to demonstrate that the thermoelastic and piezoelectric coefficients must be affected by stresses as well as temperature. In general, one should analyze Eqs. (1) and (2) together. However, due to the resulting complexity, in this work these equations are considered independently. Thus, the mutual stress dependency of the thermoelastic and piezoelectric coefficients is neglected. However, the primary effects of stresses on these coefficients are still included.

Analysis of Thermoelastic Coefficients

The engineering strains ε_{ij} and the electric field components E_m in Eqs. (1) and (2) are independent of temperature T, i.e.,

$$\frac{\partial \varepsilon_{ij}}{\partial T} = 0, \qquad \frac{\partial E_m}{\partial T} = 0 \tag{3}$$

Differentiating Eq. (1) with respect to T yields

$$\frac{\partial \sigma_{ij}}{\partial T} = \left[\frac{\partial C_{ijkl}(T)}{\partial T}\right]\varepsilon_{kl} - E_m \frac{\partial e_{ijm}(T)}{\partial T} - \frac{\partial[\lambda_{ij}(T)\cdot(T - T_0)]}{\partial T} \tag{4}$$

The thermoelastic coefficients λ_{ij} may be expressed in terms of the thermal-expansion coefficients α_{ij} as

$$\lambda_{ij}(T) = C_{ijkl}(T)\alpha_{kl}(T) \tag{5}$$

Substituting Eq. (5) into Eq. (4) and dropping the temperature dependency notation T for brevity, one obtains

$$\frac{\partial \sigma_{ij}}{\partial T} = \left(\frac{\partial C_{ijkl}}{\partial T}\right)[\varepsilon_{kl} - \alpha_{kl}(T - T_0)] - E_m \frac{\partial e_{ijm}}{\partial T} - C_{ijkl}\alpha_{kl} - C_{ijkl}(T - T_0)\frac{\partial \alpha_{kl}}{\partial T} \tag{6}$$

However, it follows from Eqs. (1) and (5) that

$$\varepsilon_{kl} - \alpha_{kl}(T - T_0) = C_{ijkl}^{-1}[\sigma_{ij} + e_{ijm}E_m] = F_{kl}(\sigma_{ij}, E_m, T) \tag{7}$$

Here C_{ijkl}^{-1} is the inverse of the elastic stiffness coefficient matrix and F_{kl} is a new function defined in Eq. (7) and affected by the stresses, the electric field, and the temperature.

Combining Eqs. (6) and (7) yields

$$\frac{\partial \sigma_{ij}}{\partial T} = \left(\frac{\partial C_{ijkl}}{\partial T}\right)F_{kl} - E_m \frac{\partial e_{ijm}}{\partial T} - C_{ijkl}\alpha_{kl} - C_{ijkl}\cdot(T - T_0)\frac{\partial \alpha_{kl}}{\partial T} \tag{8}$$

It is now necessary to analyze the energy conservation condition and the second law of thermodynamics using the approach similar to that adopted by Bert and Fu,[2] who did not include piezoelectric effects. In particular, the second law of thermodynamics requires

$$q_{i,i} = -\rho T \eta_{,t} \tag{9}$$

where q_i is the component of the heat flux in the i direction, ρ is the material density, η is the entropy density, and $(\ldots)_{,t}$ denotes $\partial(\ldots)/\partial t$.

The rate of the entropy density can be determined as[4]

$$\eta_{,t} = \frac{\partial \eta}{\partial \varepsilon_{ij}}\varepsilon_{ij,t} + \frac{\partial \eta}{\partial T}T_{,t} + \frac{\partial \eta}{\partial E_m}E_{m,t} \tag{10}$$

Received Sept. 12, 1997; revision received Sept. 8, 1998; accepted for publication Sept. 22, 1998.

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The components of the tensor of stresses, the entropy density, and the components of the electric displacements are the following functions of the electric Gibbs function χ (Ref. 4):

$$\sigma_{ij} = \rho\frac{\partial\chi}{\partial\varepsilon_{ij}}, \qquad \eta = -\frac{\partial\chi}{\partial T}, \qquad D_m = -\frac{\partial\chi}{\partial E_m} \tag{11}$$

Now Eq. (9) can be written as

$$q_{i,i} = \rho T\left(\frac{\partial^2\chi}{\partial\varepsilon_{ij}\partial T}\varepsilon_{ij,t} + \frac{\partial^2\chi}{\partial T^2}T_{,t} - p_m E_{m,t}\right) \tag{12}$$

where

$$p_m = \frac{\partial D_m}{\partial T} \tag{13}$$

is a piezoelectric constant.

Introducing the specific heat at constant strain and electric field[4]

$$c_{E,\varepsilon} = \frac{\partial\eta}{\partial T}T = -\frac{\partial^2\chi}{\partial T^2}T \tag{14}$$

the heat-flux derivative can be written as

$$q_{i,i} = -\rho c_{E,\varepsilon}T_{,t} \tag{15}$$

Combining Eqs. (11), (12), and (15), one obtains

$$q_{i,i} = -\rho c_{E,\varepsilon}T_{,t} + T\frac{\partial\sigma_{ij}}{\partial T}\varepsilon_{ij,t} - \rho T p_m E_{m,t} \tag{16}$$

Note that in the absence of the piezoelectric effect Eq. (16) is reduced to the result obtained by Bert and Fu.[2] In the case of an adiabatic process, $q_{i,i} = 0$, and combining Eqs. (8) and (16) yields

$$\rho c_E(T_{,t}/T)\varepsilon_{ij,t}^{-1} = C_{ijkl,T}F_{kl} - E_m e_{ijm,T} - C_{ijkl}\alpha_{kl} - C_{ijkl}\cdot(T - T_0)\alpha_{kl,T} - \rho T p_m E_{m,t} \tag{17a}$$

or

$$\rho c_E(T_{,t}/T)\varepsilon_{ij,t}^{-1} = C_{ijkl,t}F_{kl} - E_m e_{ijm,T} - C_{ijkl}[\alpha_{kl}\cdot(T - T_0)]_{,T} - \rho T p_m E_{m,t} \tag{17b}$$

Obviously, in the case in which all of the measurements are conducted at a constant reference temperature and in which the properties are independent of temperature, the right-hand side of Eq. (17b) becomes simply $-\lambda_{ij}(T_0)$. Thus, a relationship between the thermoelastic coefficients at the reference temperature and the thermal-expansion coefficient at temperature T is

$$C_{ijkl}\frac{\partial[(T - T_0)\alpha_{kl}]}{\partial T} = \lambda_{ij}(T_0) + R_{ij} \tag{18}$$

where

$$R_{ij} \equiv F_{kl}\left(\frac{\partial C_{ijkl}}{\partial T}\right) - E_m\frac{\partial e_{ijm}}{\partial T} - \rho T p_m E_{m,t} \tag{19}$$

Integrating Eq. (18) from the reference temperature to a current temperature, one obtains

$$(\alpha_{kl})(T - T_0) = \int_{T_0}^{T} C_{ijkl}^{-1}[\lambda_{ij}(T_0) + R_{ij}]\,\mathrm{d}T \tag{20}$$

From Eq. (20), one concludes that the thermal-expansion coefficients and, thus, the thermoelastic coefficients of a material with temperature-dependent elastic stiffnesses must depend on the stresses and the electric field as well as the temperature due to the presence of R_{ij}. The stress dependency is consistent with the experiments of Rosenfield and Averbach.[5]

Application of this theory to the one-dimensional case of a slender bar is presented in the Appendix.

Analysis of Piezoelectric Coefficients

First, it is convenient to replace Eq. (2) by

$$D_m = d_{mij}^T\sigma_{ij} + \tilde{\varepsilon}_{mk}^{\sigma,T}E_k + \left(p_m^\sigma\right)(T - T_0) \tag{21}$$

where d_{mij} is a piezoelectric tensor. The coefficients in Eq. (21) are related to those in Eqs. (1) and (2) by[6]

$$\tilde{\varepsilon}_{mk}^{\sigma,T} = \tilde{\varepsilon}_{mk}^{*} = \tilde{\varepsilon}_{mk}^{\varepsilon,T} + d_{mij}^T e_{mij}^T$$
$$p_m^{\sigma,T} = p_m^{*} = p_m^{\varepsilon,T} + d_{mij}^T\lambda_{ij}^{E,T}, \qquad e_{mij}^T = C_{ijkl}^{E,T}d_{mkl}^T \tag{22}$$

Similar to the analysis of thermoelastic coefficients, the superscripts are omitted in the subsequent discussions.

Now the following analogies are apparent between the thermoelastic and thermal-expansion coefficients on the one hand and the two kinds of piezoelectric coefficients on the other.

Thermomechanical:

$$\sigma_{ij} = -\lambda_{ij}\cdot(T - T_0), \qquad \lambda_{ij} = C_{ijkl}\alpha_{kl}$$

Electromechanical:

$$D_m = e_{mij}\varepsilon_{ij}, \qquad e_{mij} = C_{ijkl}d_{mkl} \tag{23}$$

Taking the electric-displacement field to be independent of T and differentiating Eq. (21), one obtains

$$\sigma_{ij}\frac{\partial d_{mij}}{\partial T} + d_{mij}\frac{\partial\sigma_{ij}}{\partial T} + R_m = 0 \tag{24}$$

where

$$R_m = E_k\frac{\partial\tilde{\varepsilon}_{mk}^{*}}{\partial T} + \frac{\partial\left[(T - T_0)p_m^{*}\right]}{\partial T} \tag{25}$$

Equation (24) may be rewritten as

$$\frac{\partial\sigma_{ij}}{\partial T} = -d_{mij}^{-1}\left[\sigma_{ij}\left(\frac{\partial d_{mij}}{\partial T}\right) + R_m\right] \tag{26}$$

where d_{mij}^{-1} is the inverse of the d_{mij} matrix.

If one considers the case in which the material properties are measured at the reference temperature and assumed to be independent of temperature, Eq. (26) may be reduced to simply

$$\frac{\partial\sigma_{ij}}{\partial T} = -[d_{mij}(T_0)]^{-1}p_m^{*}(T_0) \tag{27}$$

Now equating the right-hand sides of Eqs. (26) and (27), one obtains

$$d_{mij}^{-1}\left[\sigma_{ij}\left(\frac{\partial d_{mij}}{\partial T}\right) + R_m\right] = [d_{mij}(T_0)]^{-1}p_m(T_0) \tag{28}$$

Noting that Eq. (28) contains the stresses, it is clear that, if the piezoelectric coefficients depend on temperature (i.e., $\partial d_{mij}/\partial T \neq 0$), then they also must depend on the stresses and the electric field (through R_m).

Krueger[7] measured the effect of stress on the piezoelectric coefficient. However, insufficient data exist to verify the present theory in a quantitative way.

Concluding Remarks

A mathematical model for the thermoelectroelastic behavior of a solid with temperature-dependent elastic stiffnesses and piezoelectric coefficients has been developed from first principles. The results show that the thermal-expansion coefficients and the piezoelectric coefficients must depend on the stresses. This has been verified experimentally for the former case, but no pertinent experimental results are available to verify or to refute the latter.

Appendix: One-Dimensional Case

The axial stress in a slender, piezothermoelastic bar made of an orthorhombic crystal of class mm2 (Ref. 8) can be expressed as

$$\sigma_1 = Q_{11}\varepsilon_1 - e_{31}E_3 - \lambda_1 \cdot (T - T_0) \tag{A1}$$

where 1 and 3 are, respectively, the axial and thickness coordinates; Q_{11} is the axial elastic stiffness; and e_{31} is the only nonzero piezoelectric coefficient. The thermoelastic coefficient is given by

$$\lambda_1 = Q_{11}\alpha_1 \tag{A2}$$

where α_1 is the axial thermal-expansion coefficient.

Differentiating Eq. (A1) and using transformations similar to those described in the section Analysis of Thermoelastic Coefficients yields

$$\frac{\partial \sigma_1}{\partial T} = F(\sigma_1, E_3, T)\left(\frac{\partial Q_{11}}{\partial T}\right) - E_3\frac{\partial e_{31}}{\partial T} - Q_{11}\frac{\partial[(T - T_0)\alpha_1]}{\partial T} \tag{A3}$$

where

$$F(\sigma_1, E_3, T) \equiv \frac{\sigma_1 + e_{31}E_3}{Q_{11}} \tag{A4}$$

Finally, the counterpart of Eq. (20) is found to be

$$(\alpha_1)(T - T_0) = \int_{T0_0}^{T} \left\{\left[Q_{11}(T_0)\alpha_1(T_0) + F\left(\frac{\partial Q_{11}}{\partial T}\right) - E_3\frac{\partial e_{31}}{\partial T} - \rho T p_3 \frac{\partial E_3}{\partial t}\right] \Big/ Q_{11}(T)\right\} \mathrm{d}T \tag{A5}$$

Because $F = F(\sigma_1, E_3, T)$, it follows that the axial stress affects the axial thermal-expansion coefficient at temperature different from the reference value provided that the stiffness depends on temperature. It is well known that for typical piezoelectric materials[9] Q_{11} does vary with T.

Also, it is noted that, if $E_3 \neq 0$ and e_{31} depends on T, the thermal-expansion coefficient depends on E_3 as well.

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Parametric Resonance of Cylindrical Shells by Different Shell Theories

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I. Introduction

THE dynamic stability of thin, isotropic cylindrical shells under combined static and periodic axial forces is studied using three common thin-shell theories, namely, Donnell's,[1] Love's,[2] and Flügge's[3] shell theories. A main feature of this work is that, for each shell theory, the contribution of the stresses due to the external forces is accounted for according to the assumptions made in that particular shell theory. This is an extension of Ref. 4, in which consideration for the axial loading was based only on Donnell's[1] theory.

Studies of buckling of thin-walled isotropic cylinders under axial compression, torsional loadings, bending, hydrostatic pressure, and lateral pressure have been extensively covered. However, structural components under periodic loads can undergo parametric resonance that may occur over a range of forcing frequencies, and if the load is compressive to the structure, resonance or instability can and usually does occur even if the magnitude of the load is below the critical buckling load of the structure. It is thus of prime importance to investigate the dynamic stability of dynamic systems under periodic loads. The parametric resonance of cylindrical shells under axial loads has become a popular subject of study. It was first examined by Bolotin,[5] Yao,[6] and Vijayaraghavan and Evan-Iwanowski.[7] For thin cylindrical shells under periodic axial loads, the method of solution is almost always first to reduce the equations of motion to a system of Mathieu–Hill equations. The dynamic stability for such a system of equations can then be analyzed by a number of methods.

In the present analysis, the dynamic stability of thin, isotropic cylindrical shells under combined static and periodic axial forces is studied using three different shell theories: Donnell's,[1] Love's,[2] and Flügge's.[3] The treatment of the stresses due to the external loadings is done based on the assumptions made in that shell theory. A normal-mode expansion yields a system of Mathieu–Hill equations, and the parametric resonance response is analyzed based on Bolotin's[5] method. The present formulation of the problem is also made general to accommodate any boundary conditions, but for reasons of simplicity, the comparison study is carried out only for the case of simply supported boundary conditions. Numerical results of the instability regions are presented and are compared with those of Ref. 4.

II. Theory and Formulation

The cylindrical shell considered is as in Ref. 4, a thin, uniform shell of length L, thickness h, and radius R. The extensional pulsating axial load is given by

$$N_a(x, t) = N_o + N_s \cos Pt \tag{1}$$

where P is the frequency of excitation in radians per unit time.

In the present analysis, three shell theories for a thin-walled cylindrical shell are compared. They are Donnell's,[1] Love's,[2] and Flügge's[3] theories for thin cylindrical shells. The theoretical formulation follows that of Ref. 4, but the consideration for the dynamic

Received Aug. 20, 1997; revision received July 1, 1998; accepted for publication Aug. 24, 1998.

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